

# DETECTION PERFORMANCE OF M-ARY RELAY TREES WITH NON-BINARY MESSAGE ALPHABETS

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## ABSTRACT

We study the detection performance of  $M$ -ary relay trees, where only the leaves of the tree represent sensors making measurements. The root of the tree represents the fusion center which makes an overall detection decision. Each of the other nodes is a relay node which aggregates  $M$  messages sent by its child nodes into a new compressed message and sends the message to its parent node. Building on previous work on the detection performance of  $M$ -ary relay trees with binary messages, in this paper we study the case of non-binary relay message alphabets. We characterize the exponent of the error probability with respect to the message alphabet size  $\mathcal{D}$ , showing how the detection performance increases with  $\mathcal{D}$ . Our method involves reducing a tree with non-binary relay messages into an equivalent higher-degree tree with only binary messages.

**Index Terms**—  $M$ -ary relay trees, distributed detection, message alphabet.

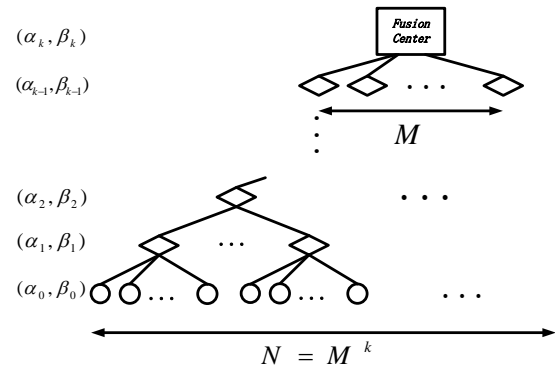
## 1. INTRODUCTION

Consider the *distributed detection* problem introduced in [1]: each sensor makes a measurement and sends a compressed version of it to the fusion center, possibly through intermediate relay nodes. This problem has been well studied in context of the parallel configuration where each sensor connects to the fusion center directly [1]–[3]. In this case, the detection error probability decays exponentially fast with respect to the number of sensors  $N$ . More general configurations also have been considered, including the tandem configuration [4],[5], and the bounded-height tree configuration [6]. In the tandem case, the error probability decays sub-exponentially [5]. In the bounded-height tree case, the error probability decays exponentially fast with an optimal decay exponent which equals that of the parallel configuration [6].

The detection performance of trees with unbounded heights has been considered in [7]–[11]. In [7], we show

that in balanced binary relay trees the total error probability decays sub-exponentially fast with exponent  $\sqrt{N}$ . In [8] and [9], we consider balanced binary relay trees with sensor failures and communication link failures. We show that, compared with the non-failure case, the decay exponent scaling law remains unchanged in the sensor failure case. Moreover, the scaling law also remains unchanged in the link failure case if the link failure probabilities decay to 0 sufficiently fast.

The  $M$ -ary relay tree configuration has been considered in [10] and [11]. As shown in Fig. 1, leaf nodes are sensors undertaking independent measurements of the same event. Only the leaves are sensors making measurements in this tree architecture. These measurements are compressed into binary messages and forwarded to the parent nodes at the next level. Each non-leaf node with the exception of the root, the fusion center, is a relay node, which combines  $M$  binary messages into one new binary message and forwards the new binary message to its parent node. This process takes place at each node, culminating at the fusion center at which the final decision is made based on the information received.



**Fig. 1.** An  $M$ -ary relay tree with height  $k$ . Circles represent sensors making measurements. Diamonds represent relay nodes which fuse  $M$  binary messages. The rectangle at the root represents the fusion center making an overall decision.

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In the case where all the nodes in the tree transmit binary messages upward, the optimal decay exponent for the total error probability  $P_N$  at the fusion center has been derived for  $M$ -ary relay trees in [10]:

$$\log_2 P_N^{-1} = O(N^{\log_M \frac{(M+1)}{2}}).$$

We show in [11] that the majority dominance rule is the optimal fusion rule in the case where  $M$  is odd and it is sub-optimal in the case where  $M$  is even. More precisely, the decay rate of the total error probability with the majority dominance fusion rule is

$$\log_2 P_N^{-1} = \Theta(N^{\log_M \lfloor \frac{(M+1)}{2} \rfloor}).$$

In [10], the case where non-binary message alphabets are allowed in  $M$ -ary relay trees is considered. Suppose that all nodes in the tree with the exception of the fusion center are allowed to transmit messages from message alphabets with size  $m$ . Then with the scheme in [10],

$$\log_2 P_N^{-1} = \Omega(N^\rho),$$

where  $\rho = 1 + \ln(1 - 1/m)/\ln M$ .

In this paper, we introduce another message-passing scheme for  $M$ -ary relay trees with non-binary message alphabets. We show how the decay exponent increases with respect to the size of the non-binary message alphabet. Compared with the scheme in [10], we show that in order to achieve the same decay exponent, our scheme involves much lower average message sizes.

## 2. MAIN RESULTS

Consider the binary hypothesis testing problem in  $M$ -ary relay trees. We characterize the detection performance by looking at the total error probability  $P_N$  at the fusion center. We have derived in [11] the decay rate of the total error probability at the fusion center in the case where relay messages are all binary, that is,

$$\log_2 P_N^{-1} = \Theta(N^{\log_M \lfloor \frac{(M+1)}{2} \rfloor}).$$

In this paper, we allow more general message alphabet (non-binary) with size  $\mathcal{D}$ , and we denote this tree by  $(M, \mathcal{D})$ -tree. We have studied the detection performance of  $(M, 2)$ -trees in [11] by investigating how fast the total error probability decays to 0. What about the detection performance when  $\mathcal{D}$  is an arbitrary finite integer?

We denote by  $u_o^k$  the output message for each node at the  $k$ -th level after fusing  $M$  input messages  $\mathbf{u}_i^{k-1} = \{u_1^{k-1}, u_2^{k-1}, \dots, u_M^{k-1}\}$  from its child nodes at the  $(k-1)$ -th level, where  $u_j^{k-1} \in \{0, 1, \dots, \mathcal{D}\}$  for all  $j \in \{1, 2, \dots, M\}$ . First, we consider an  $(M, \mathcal{D})$ -tree with height  $k_0$ , in which

there are  $M^{k_0}$  sensors. We assume that the message alphabet size is sufficiently large; more precisely,

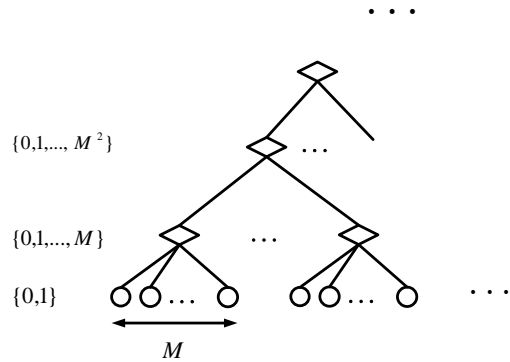
$$\mathcal{D} \geq M^{k_0-1} + 1. \quad (1)$$

Suppose that each sensor compresses its measurement into a binary message  $u_o^0 \in \{0, 1\}$  and sends it upward to its parent node. Moreover, each relay node simply sums up the messages it receives from its immediate child nodes and sends the summation to its parent node; that is,

$$u_o^k = \sum_{t=1}^M u_t^{k-1}.$$

Then we can show that the output message for each node at the  $k$ -th level is an integer from  $\{0, 1, \dots, M^k\}$  for all  $k \in \{0, 1, \dots, k_0 - 1\}$ . Moreover, this message essentially represents the number of its own child sensors that send ‘1’ upward. (A *child sensor* of a node in the tree is any leaf node (sensor) attached to the subtree rooted at that node.)

Because of inequality (1), at each level  $k$  in the tree, the message alphabet size  $\mathcal{D}$  is large enough to represent all possible values of  $u_o^k$  ( $k \in \{0, \dots, k_0\}$ ). In particular, the fusion center (at level  $k_0$ ) knows the total number of sensors that send ‘1’ upward. In this case, the detection performance is the same as a parallel configuration, where each sensor sends a binary message to the fusion center directly. Recall that in the parallel configurations, the total error probability decays exponentially fast to 0.



**Fig. 2.** A message-passing scheme for non-binary message alphabets in  $M$ -ary relay tree.

Next we consider the case where tree height is very large. As shown in Fig. 2, we apply the scheme described above, that is, the sensors send binary compressions of their measurements upward to their parent nodes. Moreover, each relay node simply sends the sum of the messages received to the parent node; i.e.,

$$u_o^k = \sum_{t=1}^M u_t^{k-1}. \quad (2)$$

From the assumption of large tree height, it is easy to see that the message alphabet size is not large enough for all the relay nodes to use the fusion rule described in (2). With some abuse of notation, we let  $k_0$  to be the integer  $k_0 = \lceil \log_M(\mathcal{D} - 1) \rceil$ . Note that

$$M^{k_0-1} + 1 \leq \mathcal{D} \leq M^{k_0} + 1. \quad (3)$$

From the previous analysis, we can show that with this scheme the nodes at the  $k_0$ -th level knows the number of '1's from its child sensors. Therefore, it is equivalent to consider the case where the nodes at level the  $k_0$  connect to  $M^{k_0}$  sensors directly (all the intermediate relay nodes can be ignored). However, we cannot use the fusion rule described in (2) for the nodes at  $k_0$ -th level to generate the output messages because the message alphabet size is not large enough. Hence, we let each node at level  $k_0$  to aggregate the  $M$  messages from its immediate child nodes into a new binary message using the majority dominance rule (with random tie-breaking; same fusion rule as in [11]). Therefore, the output message from each node at the  $k_0$ -th level is binary again. We can simply apply the fusion rule (2) and repeat this process throughout the tree, culminating at the fusion center.

**Theorem 1.** *The detection performance of  $(M, \mathcal{D})$ -trees is equal to that of  $(M^{k_0}, 2)$ -trees, where  $k_0 = \lceil \log_M(\mathcal{D} - 1) \rceil$ . In particular, if  $P_N$  be the total error probability at the fusion center for  $(M, \mathcal{D})$ -tree, then*

$$\log_2 P_N^{-1} = \Theta(N^\varrho),$$

where

$$\varrho := \begin{cases} \frac{\ln(M^{k_0}+1)}{\ln M^{k_0}} - \frac{\log_M 2}{k_0}, & \text{if } M \text{ is odd,} \\ 1 - \frac{\log_M 2}{k_0}, & \text{if } M \text{ is even.} \end{cases}$$

*Proof.* Consider an  $(M, \mathcal{D})$ -tree with the scheme described above. It is easy to see that equivalently we can consider a tree where the sensors connect to the nodes at the  $k_0$ -th level directly. In addition, because of the recursive strategy applied throughout the tree, it suffices to consider the tree where the nodes at the  $\ell k_0$ -th level connect to the nodes at the  $(\ell+1)k_0$ -th level directly for all non-negative integers  $\ell$ . Therefore, the detection performance of  $(M, \mathcal{D})$ -tree is equal to that of the corresponding  $(M^{k_0}, 2)$ -tree.

We have shown in [11] that the total error probability in  $(M, 2)$ -trees decays to 0 with the following rate:

$$\log_2 P_N^{-1} = \Theta(N^{\lceil \log_M \lfloor \frac{M+1}{2} \rfloor \rceil}).$$

Therefore, the decay rate for  $(M^{k_0}, 2)$ -trees is simply:

$$\log_2 P_N^{-1} = \Theta(N^{\lceil \log_M k_0 \lfloor \frac{M^{k_0}+1}{2} \rfloor \rceil}),$$

which can be simplified easily as follows:

$$\log_2 P_N^{-1} = \Theta(N^\varrho),$$

where

$$\varrho := \begin{cases} \frac{\ln(M^{k_0}+1)}{\ln M^{k_0}} - \frac{\log_M 2}{k_0}, & \text{if } M \text{ is odd,} \\ 1 - \frac{\log_M 2}{k_0}, & \text{if } M \text{ is even.} \end{cases}$$

□

Notice that  $\lim_{M \rightarrow \infty} \ln(M^{k_0} + 1) / \ln M^{k_0} = 1$ , which means that the even and odd cases in the expression for  $\varrho$  are similar. Hence in the following context, we will simply analyze the case where  $M$  is even. From Theorem 1, we can see that, with larger message alphabet size, the total error probability decays more quickly. However, the change of the decay exponent is not significant because  $k_0$  depend on  $\mathcal{D}$  logarithmically. Furthermore, if  $M$  is large, then the change of the performance becomes less sensitive to the increase in  $\mathcal{D}$ .

### 3. SCHEME COMPARISON

In this section, we compare our scheme to that of [10]. We show that in order to achieve the same decay exponent, the average message size used in our scheme is much smaller than that used in [10].

First, notice that the result in [10] is a lower bound for the decay rate. On the other hand, our result contains the explicit decay rate of the total error probability using our scheme. The decay exponent in [10] is  $N^\rho$ , where  $\rho = 1 - \log_M(m/(m-1))$ . The Taylor expansion for  $\rho$  as  $m \rightarrow \infty$  is

$$\rho = 1 - \frac{1}{\ln M} \left( \frac{1}{m} + \frac{1}{2m^2} + \frac{1}{3m^3} + O\left(\frac{1}{m^4}\right) \right).$$

Therefore, for large  $m$  we have

$$\rho < 1 - \frac{1}{\ln M} \left( \frac{1}{m} \right).$$

On the other hand, the decay exponent associated with our scheme is  $N^\varrho$ , where

$$\varrho = 1 - \frac{\log_M 2}{k_0} = 1 - \frac{1}{\log_2 M} \left( \frac{1}{k_0} \right).$$

Therefore, in order to achieve the same decay exponent in the asymptotic sense, we should have

$$m = \Theta(k_0).$$

Notice that in our scheme the maximum alphabet size required is  $M^{k_0-1} + 1$ . Therefore, the maximum alphabet size in our scheme is much larger than that of the scheme in [10]. However, all the nodes in [10] use the same message alphabet with size  $m$ . In our scheme only very few nodes in the tree are essentially using the maximum message alphabet. For example, the sensors only send binary messages upward to their parent nodes.

It is interesting to compare the average message size used in order to achieve such detection performance. For the scheme in [10], the average message size used (in bits) is simply  $b = \log_2 m = \Theta(\log_2 k_0)$ . On the other hand, the average size in bits used in our scheme can be calculated as follows:

$$\bar{b}(k_0) = \frac{M^{k_0} + \dots + M \log_2(M^{k_0-1} + 1)}{M^{k_0} + M^{k_0-1} + \dots + M}$$

We have

$$\log_2(M^k + 1) > \log_2 M^k = k \log_2 M$$

and

$$\log_2(M^k + 1) < \log_2(2M^k) = 1 + k \log_2 M$$

for all  $k$ . Therefore, the average size in bits is lower bounded by the following inequality:

$$\begin{aligned} \bar{b}(k_0) &> \frac{M^{k_0} + M^{k_0-1} \log_2 M + \dots + M(k_0 - 1) \log_2 M}{M^{k_0} + M^{k_0-1} + \dots + M} \\ &= \frac{M^{k_0} + M^{k_0-1} + \dots + M}{M^{k_0} + M^{k_0-1} + \dots + M} \\ &\quad + \frac{\log_2 M (M^2(M^{k_0-1} - 1) - M(k_0 - 1)) / (M - 1)^2}{M^{k_0} + M^{k_0-1} + \dots + M} \\ &= \frac{M^{k_0} - M^{k_0-1}}{M^{k_0} - 1} + \frac{M \log_2 M}{M - 1} \frac{M^{k_0-1} - 1 - M(k_0 - 1)}{M^{k_0} - 1}. \end{aligned}$$

In addition, it is upper bounded by:

$$\bar{b}(k_0) < 1 + \frac{M \log_2 M}{M - 1} \frac{M^{k_0-1} - 1 - M(k_0 - 1)}{M^{k_0} - 1}.$$

From these inequalities, it is easy to show that as  $k_0 \rightarrow \infty$ ,

$$\bar{b}(k_0) \nearrow 1 - \frac{1}{M} + \frac{\log_2 M}{M}.$$

Therefore, with large  $k_0$ , the average message size in terms of bits in our scheme is much smaller than that in [10].

$$b = \log_2 m = \Theta(k_0) \gg 1 - \frac{1}{M} + \frac{\log_2 M}{M} \geq \bar{b}(k_0).$$

#### 4. CONCLUSION

We have studied the detection performance of  $M$ -ary relay trees with non-binary message alphabet. We provide a message-passing scheme that achieves better performance than that with binary message alphabet. We show quantitatively how the decay exponent for the total error probability increases with the message alphabet size. In addition, we compare our scheme with that in [10] and find that for the same decay rate, our scheme uses much smaller messages on average.

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